Math 421 — Midterm Exam

October 20, 2022

Name: Solutions

1. **Do not open this exam until you are told to do so.**
2. This exam has 6 pages including this cover. There are 5 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. You may use no aids (e.g., calculators or notecards) on this exam.
4. In your solutions, you may refer to any of the theorems proved in class or on the homework.

Problem	Points	Score
1	5	
2	5	
3	6	
4	5	
5	4	
Total	25	

1. [5 points] For each part **a. - e.**, give **an example** of a group G satisfying the given property. You don't need to justify your answer.

a. [1 point] G is an abelian group that's not cyclic.

$\mathbb{Z}_2 \times \mathbb{Z}_2$

b. [1 point] G has infinite order and is not abelian.

$\mathbb{Z} \times S_3$

c. [1 point] G has more than one subgroup of order 2.

D_8

d. [1 point] G has nontrivial automorphism group.

$\mathbb{Z}_3$

e. [1 point] G has a proper nontrivial center. That is, $Z(G) \neq 1$ and $Z(G) \neq G$.

D_8

2. [5 points] Let G be a group. Let $f : G \to G$ be the function defined $f(g) = g^{-1}$. Show that f is a homomorphism if and only if G is abelian.

First assume f is a homomorphism. Let $x, y \in G$.

Then
$$\begin{aligned} xy &= f(x^{-1}) f(y^{-1}) \\ &= f(x^{-1}y^{-1}) \\ &= f((yx)^{-1}) = yx. \end{aligned}$$

Thus G is abelian.

Conversely, assume G is abelian. Then if $x, y \in G$,

$$\begin{aligned} f(xy) &= (xy)^{-1} \\ &= y^{-1}x^{-1} \\ &= x^{-1}y^{-1} = f(x)f(y). \end{aligned}$$

Thus, f is a homomorphism.

3. [6 points] Short answer. No justification necessary.

a. [2 points] Compute $((12)(345))^{-1}$ in S_5.

$$((12)(345))^{-1}$$
$$= (345)^{-1}(12)^{-1}$$
$$= (354)(12)$$

b. [2 points] Find the order of the subgroup $\langle(12),(34)\rangle \leq S_4$.

$\langle (12),(34) \rangle = \{1, (12), (34), (12)(34)\}$, which has order 4.

c. [2 points] Compute the centralizer, $C_{Q_8}(i)$, of i in the Quaternion group.

i commutes with $1, i, -i, -1$, but not with j, so the centralizer must have order 4 (Lagrange), so

$$C_{Q_8}(i) = \{1, i, -i, -1\} = \langle i \rangle.$$

4. [5 points] Let G be a group and H and K subgroups of G. Assume H is *not* a subset of K. Show that $H \cup K \leq G$ if and only if $K \subseteq H$.

First assume $H \cup K \leq G$. Let $k \in K$. Want to show $k \in H$.

Since $H \not\subseteq K$, take $h \in H \setminus K$.

Then $hk \in H$.

Since H is a subgroup, $h^{-1} \in H$, so $k = h^{-1}(hk) \in H$.

Thus, $K \subseteq H$.

Now assume $K \subseteq H$.

Then $K \cup H = H \leq G$, so we're done.

5. [4 points] For **a. - d.** circle **TRUE** or **FALSE**. You don't need to justify your answer.

a. [1 point] Every nontrivial group contains at least one cyclic subgroup.

TRUE **FALSE**

b. [1 point] Let G be a group that acts on a set A, and let $g \in G$. If $g \cdot a = a$ for all $a \in A$, then $g = 1$.

TRUE **FALSE**

c. [1 point] If a group G acts on a set A and the kernel of the action is K, then $K \leq G_a$ (the stabilizer of a) for each $a \in A$.

TRUE **FALSE**

d. [1 point] Let Z_{100} be a cyclic group of order 100 generated by x. Then $\langle x^4 \rangle = \langle x^8 \rangle$.

TRUE **FALSE**